**Numerical solution of Schrödinger equation with PT-symmetric periodic potential, and its Gamow integral**

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**Abstract**
In a number of preceding papers we introduced a new PT-symmetric periodic potential, derived from biquaternion radial Klein-Gordon equation. In the present paper we will review our preceding result, and continue with numerical solution of Gamow integral for that periodic potential. And then we also compare with other periodic potentials which are already known, such as Posch-Teller or Rosen-Morse potential. We also discuss a number of recent development in the context of condensed matter nuclear science, in particular those experiments which are carried out by Prof. A. Takahashi and his team from Kobe University. There is hint to describe his team's experiment as 'mesofusion' (or mesoscopic fusion). We then analyze possibility to enhance the performance of Takahashi's mesofusion experiment under external pulse field. Further experiments are of course recommended in order to verify or refute the propositions outlined herein.

## a. Introduction

In a number of preceding papers we introduced a new PT-symmetric periodic potential, derived from biquaternion radial Klein-Gordon equation. [1][2] In the present paper we will review our preceding result, and continue with numerical solution of Gamow integral for that periodic potential. And then we also compare with other periodic potentials which are already known, such as Posch-Teller or Rosen-Morse potential [9][10][11].

We also discuss a number of recent development in the context of condensed matter nuclear science, in particular those experiments which are carried out by Prof. A. Takahashi and his team from Kobe University [6][7]. There is hint to describe his team's experiment as 'mesofusion' (from mesoscopic fusion). We then analyze possibility to enhance the performance of Takahashi's mesofusion experiment under external pulse field.

Further experiments are recommended in order to verify or refute the propositions outlined herein.

## b. PT-symmetric periodic potential and its Gamow integral

In this section, first we will review our preceding result on the periodic potential based on radial Klein-Gordon equation, and then we discuss its numerical solution for Gamow integral.

There were some attempts in literature to introduce new type of symmetries in Quantum Mechanics, beyond the well-known CPT symmetry, chiral symmetry etc. In this regards, in recent years there are new interests on a special symmetry in physical systems, called PT-symmetry with various ramifications.

It has been argued elsewhere that it is plausible to derive a new PT-symmetric Quantum Mechanics (PT-QM) which is characterized by a PT-symmetric potential [3][4]:

$$V(x) = V(-x)\,. \tag{1}$$

One particular example of such PT-symmetric potential can be found in sinusoidal-form potential:

$$V = \sin\alpha\,. \tag{2}$$

PT-symmetric harmonic oscillator can be written accordingly [3]. Znojil has argued too [4] that condition (1) will yield Hulthen potential:

$$V(\xi) = \frac{A}{\left(1 - e^{2i\xi}\right)^2} + \frac{B}{\left(1 - e^{2i\xi}\right)}\,. \tag{3}$$

In our preceding paper [2][5], we argue that it is possible to write biquaternionic extension of Klein-Gordon equation as follows:

$$\left[\left(\frac{\partial^2}{\partial t^2} - \nabla^2\right) + i\left(\frac{\partial^2}{\partial t^2} - \nabla^2\right)\right]\varphi(x,t) = -m^2\varphi(x,t)\,, \tag{4}$$

Or this equation can be rewritten as:

$$\left(\Diamond\overline{\Diamond} + m^2\right)\varphi(x,t) = 0\,, \tag{5}$$

Provided we use this definition:

$$\begin{aligned}\Diamond = \nabla^q + i\nabla^q &= \left(-i\frac{\partial}{\partial t} + e_1\frac{\partial}{\partial x} + e_2\frac{\partial}{\partial y} + e_3\frac{\partial}{\partial z}\right) \\ &+ i\left(-i\frac{\partial}{\partial T} + e_1\frac{\partial}{\partial X} + e_2\frac{\partial}{\partial Y} + e_3\frac{\partial}{\partial Z}\right)\end{aligned} \tag{6}$$

Where $e_1$, $e_2$, $e_3$ are *quaternion imaginary units* obeying (with ordinary quaternion symbols: $e_1$=i, $e_2$=j , $e_3$ =k):

$$\begin{aligned}&i^2 = j^2 = k^2 = -1\,,\ ij = -ji = k\,, \\ &jk = -kj = i\,, ki = -ik = j\,.\end{aligned} \tag{7}$$

And quaternion *Nabla operator* is defined as [2][5]:

$$\nabla^q = -i\frac{\partial}{\partial t} + e_1\frac{\partial}{\partial x} + e_2\frac{\partial}{\partial y} + e_3\frac{\partial}{\partial z} \tag{8}$$

Note that equation (8) already included partial time-differentiation.

Therefore one can expect to use the same method described above to find solution of radial biquaternion KGE [2][5].

First, the standard Klein-Gordon equation reads:

$$\left(\frac{\partial^2}{\partial t^2} - \nabla^2\right)\varphi(x,t) = -m^2\varphi(x,t)\,. \tag{9}$$

At this point we can introduce polar coordinate by using the following transformation:

$$\nabla = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) - \frac{\ell^2}{r^2}\,. \tag{10}$$

Therefore by introducing this transformation (10) into (9) one gets (by setting $\ell = 0$ ):

$$\left(\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + m^2\right)\varphi(x,t) = 0\,. \tag{11}$$

Using similar method (10)-(11) applied to equation (5), then one gets radial solution of BQKGE for 1-dimensional condition [2][5]:

$$\left(\frac{1}{r^2}\frac{\partial}{\partial r}\left(\frac{\partial}{\partial r}\right) - \frac{1}{r^2}\frac{\partial}{\partial r}\left(\frac{\partial}{\partial r}\right) + m^2\right)\varphi(x,t) = 0\,, \tag{12}$$

Using Maxima computer package we find solution of (12) as a new potential taking the form of sinusoidal potential:

$$y = k_1 \sin\left(\frac{|m|r}{\sqrt{-i-1}}\right) + k_2 \cos\left(\frac{|m|r}{\sqrt{-i-1}}\right), \tag{13}$$

Where $k_1$ and $k_2$ are parameters to be determined. Now if we set $k_2$ =0, then we obtain the potential function in the form of PT-symmetric periodic potential (2):

$$V = k_1 \sin(\alpha)\,, \tag{14}$$

Where $\alpha = \left(\frac{|m|r}{\sqrt{-i-1}}\right)$.

In a recent paper [8], we interpret and compare this result from the viewpoint of EQPET/TSC model which has been suggested by Prof. Takahashi in order to explain some phenomena related to Condensed matter nuclear Science (CMNS).

### c. Schrödinger equation and Gamow integral of PT-symmetric periodic potential

Now let us consider a PT-Symmetric potential of the form:

$$V = k_1.\sin(\beta.r)\,, \tag{15}$$

where

$$\beta = \frac{|m|}{\sqrt{-i-1}}\,. \tag{16}$$

Hence, the respective Schrödinger equation with this potential can be written as follows:

$$\Psi''(r) = -k^2(r).\Psi(r) \tag{17}$$

Where

$$k(r) = \frac{2m}{\hbar^2}[E - V(r)] = \frac{2m}{\hbar^2}[E - k_1.\sin(b.r)] \tag{18}$$

For the purpose of finding Gamow function, in area near x=a we can choose linear approximation for Coulomb potential, such that:

$$V(x) - E = -\alpha(x - a), \tag{19}$$

Substitution to Schrödinger equation yields:

$$\Psi'' + \frac{2m\alpha}{\hbar^2}(x - a)\Psi = 0 \tag{20}$$

which can be solved by virtue of Airy function.

In principle, the Gamow function can be derived as follows:

$$\frac{d^2 y}{dx^2} + P(x)y = 0 \tag{21}$$

Separating the variables and integrating, yields:

$$\int \frac{d^2 y}{y} = \int -P(x).dx \tag{22}$$

Or

$$y.dy = \exp(-\int P(x).dx) + \mathrm{C}) \tag{23}$$

To find solution of Gamow function, therefore the integral below must be evaluated:

$$\gamma = \sqrt{\frac{2m}{\hbar^2}[V(x) - E]} \tag{24}$$

The general expression of Gamow function then is defined by:

$$\Gamma \approx \frac{1}{\eta^2} = \exp(-2\int_a^b \gamma(x)dx) \tag{25}$$

Therefore it should be clear that we can find different solutions for any given form of potential. In the present paper we will only consider a few potential, namely Takahashi's block-type potential (he called it STTBA model), and our PT-symmetric periodic potential. Rosen-Morse potential will be compared for the results only.

c.1. Takahashi's STTBA-block-type potential

For the case of Takahashi experiment [3][4][5], we can use b=5.6fm, and $r_0$=5fm, where the Gamow function is given by:

$$\Gamma = 0.218\sqrt{\mu}.\int_{r0}^{b}(V_b - E_d)^{1/2}.dr \tag{26}$$

Where he obtained $V_b$=0.256 MeV.

c.2. PT-symmetric periodic potential (14)

Here we assume that E=$V_b$=0.257MeV. Therefore the integral becomes:

$$\Gamma = 0.218\sqrt{\mu}.\int_{r0}^{b}(k_1 \sin(\beta r) - 0.257)^{1/2}.dr \tag{27}$$

By setting boundary conditions:

(a) at r=0 then Vo=-Vb—0.257 MeV
(b) at r=5.6fm then V1= $k_1 \sin(br) - 0.257$ =0.257Mev,therefore one can find estimate of m.
(c) Using this procedure solution of the equation (11) can be found.

The interpretation of this Gamow function is the tunneling rate of the fusion reaction of cluster of deuterium (with the given data) corresponding to Takahashi data, with the difference that here we consider a PT-symmetric periodic potential.

c.3. Rosen-Morse potential [8]

Another type of potential which may be considered here is known as Rosen-Morse potential [9][10]:

$$v = -2b.\cot|z| + a(a+a).\csc^2|z|, \tag{28}$$

Where z=r/d. Therefore the Gamow function can be written, respectively:

$$\Gamma = 0.218\sqrt{\mu}.\int_{r0}^{b}((-2b.\cot|z| + a(a+a).\csc^2|z|) - 0.257)^{1/2}.dr \tag{29}$$

(This section is not complete yet).

**Some new findings indicating Condensed matter nuclear science and Mesofusion**

In this section, we can mention that the most obvious objection against cold fusion is that the Coulomb wall between two nuclei makes the mentioned processes extremely unlikely to happen at low temperature. We can also mention here that there are three known reaction types in thermo fusion:

a. D+D → $^4$He+γ (23.8 MeV)
b. D+D → $^3$He+n
c. D+D → $^3$He+p

In this regards we would like to mention here some clear reasons why cold fusion cannot be analyzed in the classical framework of fission or 'thermo' fusion:

a. No gamma rays are seen;
b. The flux of energetic neutron is much lower than expected on basis of the heat production rate;
c. Lack of signature of D-D reaction;
d. Isotopes of Helium and also tritium accumulate to the Pd samples;
e. Cold fusion appears to occur more effective in Pd nano-particles [6][7];
f. The ratio of x to D atoms to Pd atoms in Pd particle must be in the critical range [0.85,0.90] for the process to occur.

Other strict experimental conditions may also be considered before we can expect repeatability of this process. In this regards, a recent experiment in Arata Hall, Osaka University, on May 22 2008 by Arata has clearly demonstrated that this process did happen. Because the experiment took place at Arata-Zhang laboratory, it then was referred to as Arata-Zhang experiment [6]. Other teams also produced excellent results, for example Prof. Takahashi and his Kobe University team [7].

The basic element of Takahashi's series of experiments is that a periodic potential of the Bloch wave type, as shown in the Figure 1 below.

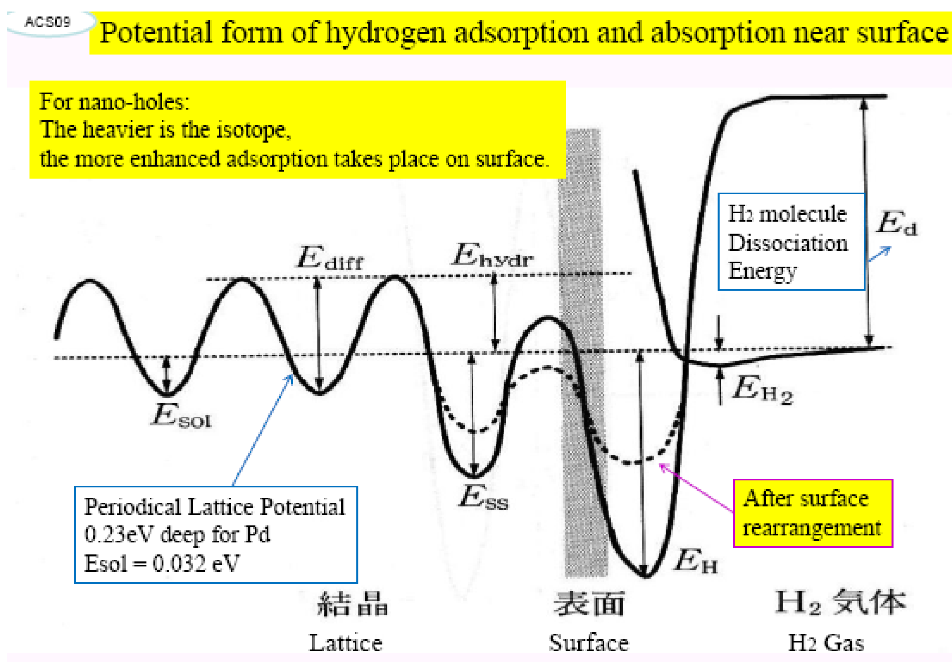


Figure 1. Lattice periodic potential used by Takahashi et al. [7]

From another line of reasoning, one can also consider this possibility of low-temperature fusion. Consider the heat production in our Earth, that some researchers consider it produced by nuclear fission or fusion. But considering that the Earth is lacking uranium (by statistical distribution), chance is that fission is unlikely, but the temperature inside the Earth is clearly much lower than the Sun, therefore the hotfusion is also unlikely to happen. Therefore apparently we can infer that inside the Earth, the heat is produced either as Condensate Nuclear transmutation (CMNS), or other types of low-energy nuclear reaction (LENR).

In other words, if we would like to keep ourselves a bit open-minded, then there other questions too which we don't find quick answer even in the natural processes surrounding us. This would mean as an indication that new types of transmutation processes should be taken into consideration as a possibility.

In this regards perhaps it would be useful to discuss a possible categorization of these new possibilities beyond standard (thermo) fusion process:

a. CANR: or chemically aided nuclear reaction, which essentially uses special types of chemical substance or enzymes [8]. For instance, see hydrino experiments (hydrino.org). Other chemists may prefer to use isoprenoids to create this new effect.
b. LENR: low-energy nuclear reaction [8], or some researchers may prefer to call it 'Lattice fusion Reaction', that is perhaps a more proper name for cold-fusion and other types of deuterium reaction which happens far below the Gamow energy. The name 'lattice fusion' also implies that the process includes neutron in some kind of solid-state physics. An indication that the fusion associated to LENR is outside the domain of standard fusion processes is lack of signature of D-D reaction, which would mean that perhaps the process is much more complicated (for instance Takahashi considered tetra-deuterium model). There is also indication of lacking of neutron emission during this process [7]. We will discuss more on these issues in subsequent section.
c. Mesofusion (or mesoscopic fusion): this belongs to experiments which can be associated to nano-Pd samples for instance by Takahashi and his team in Japan [6]. While this term is not well accepted yet, in our opinion this type of reactions will be much more common in particular for industrial applications, since nanometer devices are much more manageable rather than materials at the order of lepton or hadron scale.

**Concluding remarks: Next steps**

We would like to conclude this note with a number of some kinds of wish-list.

First of all, a rigorous theoretical framework is clearly on demand. This for instance, will include both to clarify the distinction between Mesofusion and Chromodynamics fusion, and also to consider new type of potentials.

And then, in terms of experiments it appears to be more interesting to introduce new types of tools in order to enhance the performance of these Mesofusion or Chromodynamics fusions. For instance, perhaps it would be interesting to see whether the performance can be improved by introducing either laser or external electromagnetic pulse, just like what has been done in the conventional thermo fusion.

All of these remarks are written here to emphasize that based on recent publication [5]-[8], we are clearly in the beginning of observing new types of fusion technologies, by harnessing our knowledge of hadron and chromodynamics theory.

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